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Stochastic bias in multi-dimensional excursion set approaches

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ABSTRACT

We describe a simple fully analytic model of the excursion set approach associated with two Gaussian random walks: the first walk represents the initial overdensity around a protohalo, and the second is a crude way of allowing for other factors which might influence halo formation. This model is richer than that based on a single walk, because it yields a distribution of heights at first crossing. We provide explicit expressions for the unconditional first crossing distribution which is usually used to model the halo mass function, the progenitor distributions from which merger rates are usually estimated, and the conditional distributions from which correlations with environment are usually estimated. These latter exhibit perhaps the simplest form of what is often called nonlocal bias, and which we prefer to call stochastic bias, since the new bias effects arise from 'hidden-variables' other than density, but these may still be defined locally. We provide explicit expressions for these new bias factors. We also provide formulae for the distribution of heights at first crossing in the unconditional and conditional cases. In contrast to the first crossing distribution, these are exact, even for moving barriers, and for walks with correlated steps. The conditional distributions yield predictions for the distribution of halo concentrations at fixed mass and formation redshift. They also exhibit assembly bias like effects, even when the steps in the walks themselves are uncorrelated. Our formulae show that without prior knowledge of the physical origin of the second walk, the naive estimate of the critical density required for halo formation which is based on the statistics of the first crossing distribution will be larger than that based on the statistical distribution of walk heights at first crossing; both will be biased low compared to the value associated with the physics. Finally, we show how the predictions are modified if we add the requirement that halos form around peaks: these depend on whether the peaks constraint is applied to a combination of the overdensity and the other variable, or to the overdensity alone. Our results demonstrate the power of requiring models to reproduce not just halo counts but the distribution of overdensities at fixed protohalo mass as well.

Key words: large-scale structure of Universe

1 INTRODUCTION

The excursion set approach, pioneered by Epstein (1983) and developed substantially by Bond et al. (1991), Lacey & Cole (1993), Mo & White (1996) and Sheth (1998) yields important insight into various features of hierarchical clustering. Although recent work has highlighted the limitations of this approach (Paranjape & Sheth, 2012), the limitations are primarily of a quantitative rather than qualitative nature.

The approach combines the statistics of the initial density fluctuation field with the physics of spherical or

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triaxial collapse, to make predictions for the abundance of virialized objects as a function of time. This means that it provides information about merger rates, the high-redshift progenitors of objects of fixed mass at a later time, the tendency for the mass function in dense regions to be top-heavy, and hence how the spatial clustering of these objects depends on their mass.

In the spherical collapse model, the evolution of an object is determined by its own overdensity. This enters in the excursion set approach as follows. One associates a one-dimensional random walk with each position in space; this walk shows how the initial overdensity depends on the smoothing scale over which the density is averaged. The largest scale on which this walk exceeds the critical density required for spherical collapse contains a mass; this is the excursion set estimate of the mass of the object in which this particular position in space will end-up. Therefore, in this approach, the technical problem to be solved is that of the first crossing distribution of a barrier whose height may depend on the number of steps taken by the one-dimensional random walk. The statistics of the initial fluctuation field determines the ensemble of walks over which to average.

In triaxial collapse models, the evolution of an object is determined by more than its initial overdensity (Bond & Myers, 1996; Sheth et al., 2001). In the context of such models, it is natural to ask how these extra parameters enter the excursion set approach. It should come as no surprise that each additional variable simply adds an extra walk (Sheth et al., 2001; Chiueh & Lee, 2001; Sheth & Tormen, 2002), but there is no guarantee that these variables are Gaussian distributed. As a result, the technical problem becomes one of first crossing a multi-dimensional barrier by multi-dimensional walks. However, it has recently been realized that this has non-trivial, qualitatively different, consequences for halo bias: in effect, the correlations between these other parameters on the large scale density field introduce what are known as nonlocal bias effects (Sheth et al., 2012). In this respect, the multi-dimensional excursion set approach is considerably richer than the one-dimensional one.

The main goal of this paper is to illustrate a number of these qualitatively new features of the multi-dimensional excursion set approach. Our goal here is not so much to develop a model which reproduces effects seen in simulations, as to develop insight: therefore, the emphasis is on developing a fully analytic model in which it is easy to see the origin of these new effects. It turns out that this model may not be that unrealistic – this is explored further in Achitouv et al. (2013).

Section 2 describes our model and provides expressions for the usual excursion set approach quantities, as well as for the qualitatively new ones. Section 3 describes a number of extensions, including an explicit calculation of how all the predictions are modified if protohalos are identified with peaks in the initial field. We use this to demonstrate how requiring models to reproduce both halo counts as well as overdensities at fixed halo mass provides sharp constraints. A final section summarizes.

2 TWO INDEPENDENT GAUSSIAN WALKS WITH UNCORRELATED STEPS

Let δ and g both denote zero-mean Gaussian variables, with variance $\langle\delta^2\rangle \equiv s$ and $\langle g^2\rangle \equiv \beta^2 s$ respectively. When plotted as a function of s, these represent walks associated with the overdensity and the second variable which matters for collapse. We will assume that δ and g are independent: $\langle\delta g\rangle = 0$.

We will use $f(s)$ to denote the distribution of s when

$$\delta \geq \delta_c(s) + g \tag{1}$$

for the first time. We will also be interested in $p(\delta_{1\times}|s)$, the distribution of walk heights at first crossing. The excursion set ansatz assumes that the quantity $f(s)$ is related to the mass fraction in halos having mass $m(s)$ by

$$f(s)\,\mathrm{d}s = \frac{m}{\bar\rho}\,\frac{\mathrm{d}n(m)}{\mathrm{d}m}\,\mathrm{d}m, \tag{2}$$

where $\mathrm{d}n/\mathrm{d}m$ is the comoving number density of halos of mass m, and $\bar\rho$ is the comoving background density.

2.1 Rotation of coordinate system

When the inequality (1) is saturated, it defines a line in the (δ, g) plane. The clearest way to think of this problem is to change variables to ones which run parallel and perpendicular to this line. Therefore, define

$$g_- = \frac{\delta - g}{\sqrt{1+\beta^2}} \quad \text{and} \quad g_+ = \frac{\beta\delta + g/\beta}{\sqrt{1+\beta^2}}. \tag{3}$$

Notice that $\langle g_-^2\rangle = \langle g_+^2\rangle = s$, and that these variables are independent:

$$\langle g_+ g_-\rangle = \beta\langle\delta^2\rangle - \frac{\langle g^2\rangle}{\beta} = 0. \tag{4}$$

In these variables, g_- steps towards or away from the barrier, which has height $\delta_c(s)/\sqrt{1+\beta^2}$, and g_+ steps parallel to it.

For what follows, it is useful to note that

$$\delta = \frac{g_- + \beta g_+}{\sqrt{1+\beta^2}} \quad \text{and} \quad g = \beta\,\frac{g_+ - \beta g_-}{\sqrt{1+\beta^2}}. \tag{5}$$

2.2 Unconditional first crossing distribution

The independence of g_+ and g_- means that $f(s)$ depends only on g_-. Since g_- is just a one dimensional gaussian walk, and it must cross a barrier of height $\delta_c(s)/\sqrt{1+\beta^2}$, the first crossing distribution is that for a moving barrier, for which simple approximations are available (Sheth & Tormen, 2002).

For the special case in which δ_c does not depend on s, the first crossing distribution is

$$sf(s) = \frac{\nu f(\nu_\beta)}{2} = \nu\,\frac{\exp(-\nu^2/2)}{\sqrt{2\pi}}, \tag{6}$$

where

$$\nu^2 \equiv \frac{\delta_c(0)^2/s}{1+\beta^2} \equiv \nu_\beta^2. \tag{7}$$

Notice that $\beta = 0$ yields the usual one-dimensional solution.

Notice also that the factor $1 + \beta^2$ can be viewed in either of two ways. Either it rescales the barrier height (which is how it appeared in the analysis above) or it rescales the variance s. Now, the first crossing distribution $f(s)\mathrm{d}s$ is usually equated with the mass fraction in halos of mass m (equation 2). If δ_c itself is expected to be related to the physics of halo formation, then the rescaling of δ_c means that one must also understand the physics which led to $\beta \neq 0$ if one wishes to derive the value of δ_c from halo abundances. Failure to do so will lead to a misestimate of the true value of the value of δ_c which matters for the physics. If we require $\delta_c \approx 1.686$, then matching halo counts requires $(1+\beta^2)^{-1} \approx 0.7$ so $\beta \approx 0.6$ (Sheth & Tormen, 1999).

2.3 Distribution of height at first crossing

Define $\delta_{1\times}$ to be the value of δ when $g_- = \delta_c(s)/\sqrt{1+\beta^2}$. Then

$$\delta_{1\times} \equiv \frac{\delta_c(s)/\sqrt{1+\beta^2} + \beta g_+}{\sqrt{1+\beta^2}} = \frac{\delta_c(s)}{1+\beta^2} + \frac{\beta\, g_+}{\sqrt{1+\beta^2}}. \quad (8)$$

Since g_+ is just a Gaussian with zero mean and variance s (recall it is independent of g_-), the expression above shows that

$$p(\delta_{1\times}|s) = \frac{\mathrm{e}^{-(\delta_{1\times}-\mu_{1\times})^2/2\Sigma^2_{1\times}}}{\sqrt{2\pi\Sigma^2_{1\times}}} \quad (9)$$

where

$$\mu_{1\times} = \frac{\delta_c(s)}{1+\beta^2} \quad \text{and} \quad \Sigma^2_{1\times} = \frac{\beta^2}{1+\beta^2}\, s. \quad (10)$$

The limit $\beta = 0$ yields a delta-function centered on $\delta_c(s)$ as it should.

If we set $\nu_{1\times} \equiv \delta_{1\times}/\sigma$ where $\sigma^2 \equiv s$, and recall from equation (7) that $\nu_\beta \equiv (\delta_c(0)/\sigma)/\sqrt{1+\beta^2}$, then it is useful to think of the distribution above as $p(\nu_{1\times}|\nu_\beta)$, the conditional distribution of $\nu_{1\times}$ given ν_β: in this case, the expression above is the standard expression for the conditional Gaussian distribution with correlation parameter $(1+\beta^2)^{-1/2}$.

Note that equation (8), and hence equation (9) are exact even when δ_c depends on s. In this respect, the distribution of $\delta_{1\times}$ at first crossing is much simpler than is the first crossing distribution itself – it always has a Gaussian shape, with the barrier only affecting the mean value of this Gaussian.

It is also worth noting that $\langle\delta_{1\times}|s\rangle = \mu_{1\times}$ is *guaranteed* to be less than δ_c. Thus, without prior knowledge of the value of β, the statistical distribution of $\delta_{1\times}$ will lead to a misestimate of the value of δ_c which is associated with the physics. In this context, it is useful to think in terms of the distribution of differences from δ_c. If we define $\Delta_{1\times-c} \equiv \delta_{1\times} - \delta_c(s)$, then it is Gaussian distributed with mean $-\delta_c(s)\,\beta^2/(1+\beta^2)$ and variance $s\beta^2/(1+\beta^2)$. I.e., the mean is $\delta_c(s)$ times the same factor by which s is rescaled. This provides a simple operational way of determining the value of β from a measurement of $p(\Delta_{1\times-c}|s)$.

2.4 Distribution of the barrier at first crossing

Similarly, define $g_{1\times}$ to be the value of g at first crossing. Then, because $g_{1\times} \equiv \delta_{1\times} - \delta_c$, it has the same distribution as $\delta_{1\times}$, but with a shifted mean. Specifically, $p(g_{1\times})$ will be Gaussian with mean $-\delta_c(s)\,\beta^2/(1+\beta^2)$ and variance $s\beta^2/(1+\beta^2)$.

2.5 The two-barrier problem and progenitor distributions

Symmetry means that the distribution of S_1 at which

$$\delta \geq \delta_{c1} + g \quad (11)$$

for the first time, given that inequality (1) was first satisfied on scale $S_0 < S_1$, is given by equation (6) but with ν^2 replaced by

$$\nu^2_{10} = \frac{(\delta_{c1} - \delta_{c0})^2}{(S_1 - S_0)(1+\beta^2)}. \quad (12)$$

The limit $\beta = 0$ yields the usual expression for progenitor distributions associated with one-dimensional walks (e.g. Lacey & Cole, 1993).

Halo formation is often identified with the time when at least half the total mass has been assembled in pieces that are each more than μ times the final mass. For $\mu > 1/2$, there can be only one such piece so the formation time distribution is given by

$$p(\delta_{cf} \geq \delta_{c1}|M, \delta_{c0}) = \int_{\mu M}^{M} \mathrm{d}m\, \frac{M}{m}\, f(m, \delta_{c1}|M, \delta_{c0}) \quad (13)$$

(Lacey & Cole, 1993). For white noise initial conditions ($s \propto m^{-1}$) and $\mu = 1/2$ this becomes

$$p(\omega_f) = 2\omega_f\, \mathrm{erfc}(\omega_f/\sqrt{2}) \quad (14)$$

where $\omega_f = \nu_{f0}$ with ν_{f0} given by equation (12). Because ω_f includes a factor of $1 + \beta^2$, the mean formation redshift will be scaled to higher values than when $\beta = 0$. This sort of rescaling yields better agreement with measurements in simulations (Giocoli et al., 2007; Moreno, Giocoli & Sheth, 2008). See Sheth (2011) for the case $\mu < 1/2$.

2.6 Conditional distributions and correlations with environment

Similarly, the distribution of s at which inequality (1) is first satisfied, given that δ has height Δ on some scale $S < s$, but G is unconstrained (except by the requirement that $\Delta - G < \delta_{c0}$), is also given by equation (6) but with ν^2 replaced by

$$\nu^2_\Delta = \frac{(\delta_{c0} - \Delta)^2}{s(1+\beta^2) - S}. \quad (15)$$

(The Appendix provides a short derivation.) This can be thought of as subtracting from the variance $s(1+\beta^2)$ the

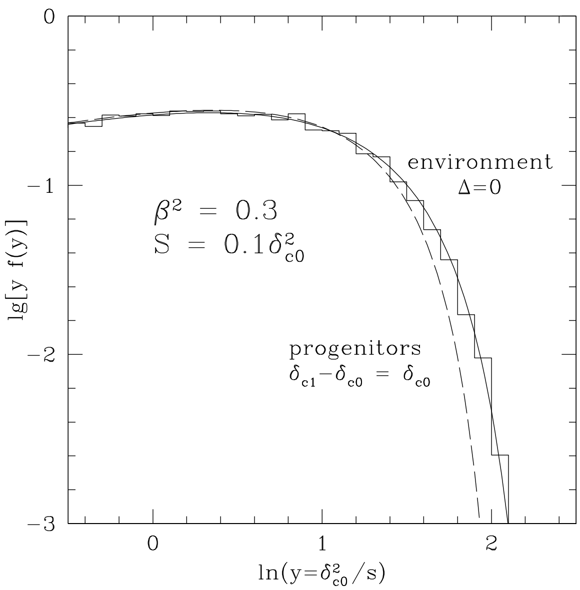

Figure 1. First crossing distributions for our two-dimensional walks with $\beta^2 = 0.3$ which are conditioned to first pass through $\Delta = 0$ on scale $S = 0.1\delta_{c0}^2$ (histogram); smooth curve shows our prediction (equation 15 in equation 6). The effective cosmology of this environment has critical density δ_{c0}; dashed curve shows the progenitor distribution with this same effective cosmology (equation 12 in equation 6). For one-dimensional walks, the solid curve would be the same as the dashed one.

piece which comes from constraining $\delta = \Delta$ on scale S, which makes its correspondence to the one-dimensional expression (the $\beta = 0$ limit of this expression) obvious.

Because equation (12) is different from (15), when expressed as a function of s rather than ν, the conditional distribution is different from the progenitor one, whereas they are the same for one-dimensional walks. The difference between the two is largest in the $s \to S$ limit, where the conditional distribution predicts more objects than does the progenitor distribution. Figure 1 illustrates. self-similar distribution. Thus, a discrepancy between the progenitor and environmental dependences of clustering provides a simple way to see if stochasticity has played a role in determining halo abundances.

Things are slightly more complicated if $\delta_c(s)$, of course, but the basic fact that progenitor and conditional distributions with $\delta_{c1} - \delta_{c0} = \delta_{c0} - \Delta$ will no longer be the same is generic.

2.7 Stochastic (nonlocal) bias

The distribution of s at which inequality (1) is first satisfied, given that the walk was at (Δ, G) on scale $S < s$, is also given by equation (6) but with ν^2 replaced by

$$\nu_{\Delta G}^2 = \frac{(\delta_{c0} - \Delta + G)^2}{(s - S)(1 + \beta^2)}. \tag{16}$$

This follows from the fact that the distance from a point (x_0, y_0) to the line $ax + by + c = 0$ is $|ax_0 + by_0 + c|/\sqrt{a^2 + b^2}$. Alternatively, one can view this as the same shift of origin to the g_- walk that is made in the one-dimensional case (e.g. Lacey & Cole, 1993). The expression above shows that G can affect halo abundances in qualitatively the same way that Δ can.

In more detail, the halo overdensity is defined by the ratio of the conditional expression to the unconditional one (Mo & White, 1996). In our case, this means that

$$1 + \delta_h(\nu|\Delta, G) = \frac{\nu_{\Delta G} f(\nu_{\Delta G})}{\nu f(\nu)}. \tag{17}$$

The peak-background split bias factors are the coefficients in the Taylor series expansion of the expression above, in the limit where $s \gg S$. If we write these as

$$1 + \delta_h \equiv \sum_{i,j} B_{ij} \frac{\Delta^i}{i!} \frac{G^j}{j!}, \tag{18}$$

then the dependence on G gives rise to what is known as nonlocal bias. Since G may also be determined by local quantities, this is, in general, a misnomer. Since it is really an effect which arises from the dependence of halo counts on the 'hidden' stochastic variable G, we think it is more accurate to call this 'stochastic' bias, which may or may not be local.

Recently, Musso et al. (2012) have shown that cross-correlating the halo overdensity field with the nth-order Hermite polynomial $H_n(\Delta/\langle\Delta^2\rangle^{1/2})$ is an efficient way of reconstructing the b_n coefficients even when $\langle\Delta^2\rangle^{1/2}$ is not small. In our case, cross-correlating with $H_i(\Delta/\langle\Delta^2\rangle^{1/2})\, H_j(G/\langle G^2\rangle^{1/2})$ yields

$$\delta_c^{i+j} B_{ij} = (-1)^j\, \nu^{i+j-1}\, H_{i+j+1}(\nu), \tag{19}$$

where $\nu^2 = (\delta_{c0}^2/s)/(1 + \beta^2)$. This reduces to the usual expression (Mo & White, 1996; Musso et al., 2012) when $j = 0$:

$$\delta_c^k B_{k0} \equiv \delta_c^k\, b_k = \nu^{k-1}\, H_{k+1}(\nu). \tag{20}$$

Since the dependence of equation (16) on G is the same as that on Δ, cross-correlating with $H_n(G/\langle G^2\rangle^{1/2})$ alone yields

$$B_{0k} \equiv c_k = (-1)^k b_k. \tag{21}$$

In this respect, the stochastic (possibly nonlocal) bias model here is simpler than that in Sheth et al. (2012), where the analogue of G was not Gaussian distributed (so the associated orthogonal polynomials were more complicated).

2.8 Assembly bias

Assembly bias is the correlation between properties of protohaloes of fixed mass and their environment, such as those first identified by Sheth & Tormen (2004), and studied since by many others. While it is generally believed that this effect should be absent in excursion set models with uncorrelated steps (White, 1996), we now show that our two-dimensional model does exhibit assembly bias, even though the steps in the walks are uncorrelated. However, we caution that we are not claiming

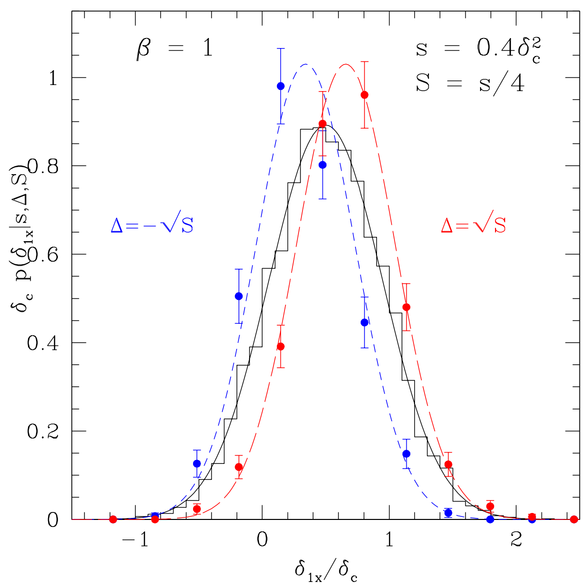

Figure 2. Dependence of walk height at first crossing, $\delta_{1\times}$, on large scale environment. Symbols with error bars show the distribution of $\delta_{1\times}$ for walks which first cross each other on scale s, and which had height Δ on scale $S < s$; smooth dashed curves show equation (22). Black histogram shows the corresponding unconditional distribution for the same value of s; smooth solid curve shows the corresponding prediction (equation 9).

that this model explains assembly bias; simply that assembly bias is part and parcel of the multi-dimensional excursion set approach, even for walks with uncorrelated steps.

The distribution of walk heights at first crossing, given that $\delta = \Delta$ on scale S, is

$$p(\delta_{1\times}|s,\Delta,S) = \frac{\mathrm{e}^{-(\delta_{1\times}-\Delta-\mu_\Delta)^2/2\Sigma_\Delta^2}}{\sqrt{2\pi\Sigma_\Delta^2}} \tag{22}$$

where

$$\mu_\Delta = \frac{\delta_c(s)-\Delta}{1+\beta^2} \quad \text{and} \quad \Sigma_\Delta^2 = \frac{(s-S)\beta^2}{1+\beta^2}. \tag{23}$$

This is the conditional analogue of equation (9).

This shows that the variance is smaller than it is for unconditioned walks, but that the difference is negligible when $s \gg S$. The mean is more interesting:

$$\langle\delta_{1\times}|s,\Delta,S\rangle = \Delta + \mu_\Delta = \frac{\delta_c(s)+\beta^2\Delta}{1+\beta^2} \tag{24}$$

is shifted by $\Delta\beta^2/(1+\beta^2)$ compared to the unconditional mean. Even more suggestively, this implies that $\langle\Delta_{1\times-c}|s,\Delta,S\rangle = [\Delta - \delta_c(s)]\,\beta^2/(1+\beta^2)$. The dependence of this mean on the larger scale Δ is this model's expression of assembly bias, and is an important way in which the two-walk problem differs from the one-walk problem. When $\beta = 0$ the distribution becomes a delta-function centered on δ_c; since it is therefore independent of Δ, this shows explicitly that the one-dimensional solution shows no assembly bias when the steps in the walk are uncorrelated.

Figure 2 illustrates the effect: objects which are surrounded by large scale overdensities tend to have larger δ_{1x} than objects of the same mass in large-scale underdensities. Since they have above average initial overdensities on scale s, they will also tend to have above average overdensities at formation (typically, on scale $\sim 2s$). The result is a correlation, at fixed halo mass, between the density at formation and environment – even though there will not be a correlation between formation time (rather than the overdensity at the formation time) and environment. (In this model, as for the one-dimensional case, any correlation between formation time and larger scale environment can only come from correlations between steps.) Since the density at formation is correlated with halo concentration at virialization (Navarro et al., 1997), our model predicts a correlation between halo concentration and environment at fixed mass.

3 EXTENSIONS

3.1 Correlated steps

Our change of variables from δ, g to walks which step parallel and perpendicular to the barrier makes it straightforward to see what should happen when both δ and g are walks with correlated steps. If the correlations are the result of smoothing with the same filter, then the unconditional distribution $f(s)$ should be replaced with the corresponding expression in Musso & Sheth (2012) (see discussion following equation 32), but the distribution of the walk height at first crossing, $p(\delta_{1\times}|s)$, remains unchanged. This is because, at first crossing, $\delta_{1\times}$ depends only on g_+ (by definition), and g_+, although it has correlated steps, is independent of g_-, so it is not constrained by the fact that $g_- = \delta_c/(1+\beta^2)$.

Following Paranjape et al. (2012) the progenitor distribution should be well-approximated by replacing $\delta_{c1} - \delta_{c0} \to \delta_{c1} - (S_\times/S)\delta_{c0}$ and $(s-S)(1+\beta^2) \to [s - (S_\times/S)^2 S](1+\beta^2)$, and the conditional distribution by replacing $\delta_{c0} - \Delta \to \delta_{c0} - (S_\times/S)\Delta$ and $s(1+\beta^2) - S \to s(1+\beta^2) - (S_\times/S)^2 S$. Still more accurate expressions follow from making the corresponding replacements in the expressions provided in Musso et al. (2012). Testing these expressions is the subject of work in progress.

3.2 Correlated Walks

Suppose instead that steps in δ are uncorrelated, whereas steps in g are correlated with those in δ. This may happen, for example, if the critical density for collapse depends on the overdensity on a larger scale, e.g. in the correlated galaxy formation model of Bower et al. (1993) or in theories of modified gravity (Lam & Li, 2012). Then, let

$$\rho \equiv \frac{\langle\delta g\rangle}{\langle\delta^2\rangle^{1/2}\,\langle g^2\rangle^{1/2}}\,, \tag{25}$$

denote the correlation parameter between δ and g. If we make the same coordinate transformation as before, then $\langle g_-^2\rangle = s(1+\beta^2-2\rho\beta)/(1+\beta^2)$, $\langle g_+^2\rangle = s(1+\beta^2+2\rho\beta)/(1+\beta^2)$ and $\langle g_+g_-\rangle = s\,\rho\,(1-\beta^2)/(1+\beta^2)$. We can always write $p(g_-,g_+) = p(g_-)\,p(g_+|g_-)$, where $p(g_+|g_-) \neq p(g_+)$ is a Gaussian distribution with mean $g_-\,\langle g_+g_-\rangle/\langle g_-^2\rangle$ and variance $\langle g_+^2\rangle\,[1-\langle g_+g_-\rangle^2/\langle g_-^2\rangle\langle g_+^2\rangle]$.

Since the first crossing distribution depends on $p(g_-)$ and not on g_+, it is given by the same expression as for uncorrelated walks, but with $\nu^2 = (\delta_c^2/s)/(1+\beta^2+2\rho\beta)$. This shows that the amount by which δ_c appears to be rescaled depends on β as well as the correlation parameter.

However, $p(\delta_{1\times}|s)$ will be affected. Namely, at first crossing, $\delta_{1\times}$ is given by equation (8), so

$$\langle\delta_{1\times}|s\rangle = \frac{\delta_c(s)}{1+\beta^2} + \frac{\beta\langle g_+|g_-\rangle}{\sqrt{1+\beta^2}} \tag{26}$$

where, because $g_- = \delta_c/(1+\beta^2)$ at first crossing,

$$\langle g_+|g_-\rangle = \frac{\delta_c(s)}{1+\beta^2}\,\rho\,\frac{1-\beta^2}{1+\beta^2-2\rho\beta}\,. \tag{27}$$

Therefore, $p(\delta_{1\times}|s)$ is Gaussian with mean and variance

$$\mu = \frac{\delta_c}{1+\beta^2}\left(1+\rho\,\frac{\beta\,(1-\beta^2)}{1+\beta^2-2\beta\rho}\right) \tag{28}$$

$$\Sigma^2 = s\,\frac{\beta^2\,(1-\rho^2)}{(1+\beta^2-2\beta\rho)}. \tag{29}$$

For $\rho=0$, this reduces to equation (9); for $\rho=1$ or -1, corresponding to complete correlation or anti-correlation, the distribution becomes a Dirac delta function centered on $\mu = 1/(1-\beta)$ or $1/(1+\beta)$, respectively. (This can be understood simply from the fact that, in these limiting cases, the two-dimensional walk is confined to a line, and this line can only cross the line defined by the barrier at a single point.)

It is a curious fact that when $\beta = 1$ (the two walks have the same variance), then there is no shift to the mean, and the variance becomes $s\,(1+\rho)/2$. This can be traced back to the fact that, when $\beta=1$, then $\langle g_+g_-\rangle = 0$; i.e., the walks in g_- and g_+ are independent (even though δ and b are correlated), but they have different variances.

But in general, correlations between the walks lead to a shift in the mean and a rescaling of the variance. However, they do not change the fact that $p(\delta_{1x}|\sigma)$ is Gaussian. In practice, one should be able to determine if $\rho \neq 0$ because the three unknowns, δ_c, β and ρ can be determined from our expressions for the mean and variance of $p(\delta_{1x}|\sigma)$ and the required rescaling of s in the first crossing distribution $f(s)$.

3.3 Higher-dimensional walks and/or other distributions

Our fundamental assumption, that equation (1) accurately captures the physics of collapse, is, of course, only an idealization. Note, however, that if other variables also mattered, and they were also Gaussian distributed, such that equation (1) becomes

$$\delta \geq \delta_c(s) + \sum_{i=1}^{n-1} g_i, \tag{30}$$

then, because the sum of Gaussians is itself Gaussian, this n-dimensional model reduces to the 2-dimensional one we have just solved, with $\beta^2 = \sum_{i=1}^{n-1}\beta_i^2$.

Alternatively, suppose instead that

$$\delta \geq \delta_c + \chi, \tag{31}$$

where χ follows a non-Gaussian distribution. E.g., Sheth et al. (2012) study a model in which δ_c is independent of s, but χ^2 is drawn from a chi-squared distribution with five degrees of freedom. However, this distribution has a mean which depends on s. If the distribution of $\Delta\chi \equiv \chi - \langle\chi\rangle$ is not too different from a Gaussian, then we can use our 2-dimensional Gaussian model as a reasonable approximation to this one, with $\delta_c(s)$ in equation (1) equal to $\delta_c + \langle\chi\rangle$ and g a zero-mean Gaussian variate having the same variance as $\Delta\chi$. E.g., for the model in Sheth et al. (2012), $\langle\chi\rangle \approx 0.95\sqrt{s}$ and $\langle(\Delta\chi)^2\rangle \approx 0.09s$. I.e., this model should be reasonably well approximated by our two-Gaussian model with $\delta_c(s) = \delta_c + 0.95\sqrt{s}$ and $\beta^2 = 0.09$.

This has the following interesting consequence. At first crossing, the distribution of $\Delta\chi$ will be like that of $g_{1\times}$, meaning that it should have mean and variance approximately given by $-\delta_c(s)\beta^2/(1+\beta^2)$ and $s\,\beta^2/(1+\beta^2)$. Since the variance of the initial variate χ was $\beta^2 s$, one should think of $\chi_{1\times}$ as having variance reduced by $(1+\beta^2)^{-1}$. For it to still have approximately the same functional form as χ itself, it should have mean $0.95\sqrt{s/(1+\beta^2)}$, which is smaller than the original value of $0.95\sqrt{s}$. For $\beta \ll 1$, we can think of this as a shift in the mean by $-0.95\sqrt{s}\beta^2/2$. The actual shift, $-\delta_c(s)\beta^2/(1+\beta^2)$, has the same sign, but a different amplitude, indicating that the distribution of $\chi_{1\times}$ will not be quite the same as that of χ itself.

We end this discussion with a word of caution: Although mapping to an effective Gaussian is useful, it may hide interesting physics. For example, the non-Gaussian stochasticity in Sheth et al. (2012) results in a quadrupolar signature for Lagrangian space halo bias; using an effective Gaussian obscures the origin of this angular dependence.

3.4 Excursion set peaks

For walks associated with peaks in $\delta - g$, one must simply add a weight which depends on $\mathrm{d}(\delta-g)/\mathrm{d}s$ (Musso & Sheth, 2012). The associated first crossing distribution becomes that for excursion set peaks (Paranjape & Sheth, 2012), provided we remember to rescale $\delta_c(s) \to \delta_c(s)/\sqrt{1+\beta^2}$, because the peaks are in g_- rather than in δ. Namely,

$$sf(s) = \frac{\exp(-\nu_\beta^2/2)}{2\gamma\sqrt{2\pi}}\int \mathrm{d}x\,x\,p(x|\gamma\nu_\beta)\,\frac{\mathcal{F}(x)}{(R_*/R)^3} \tag{32}$$

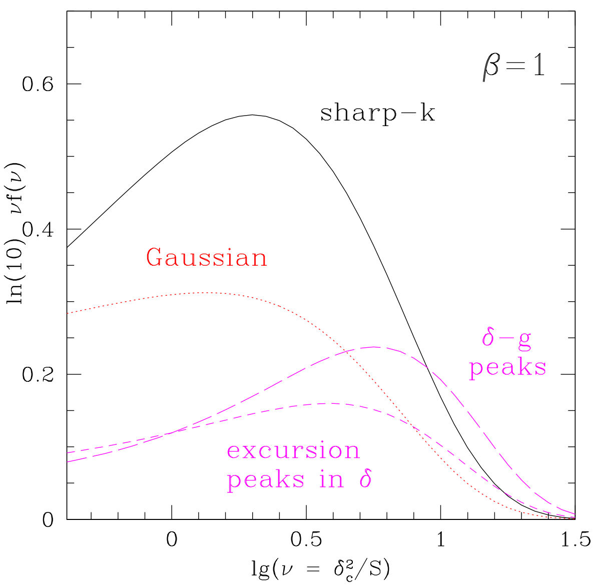

Figure 3. First crossing distribution for all walks when steps are uncorrelated (solid); when steps are correlated because of Gaussian smoothing and the power-spectrum is $P(k) \propto k^{-1.2}$; when the walks are centered on peaks in $\delta - g$ (equation 32) and on peaks in δ only (equation 34).

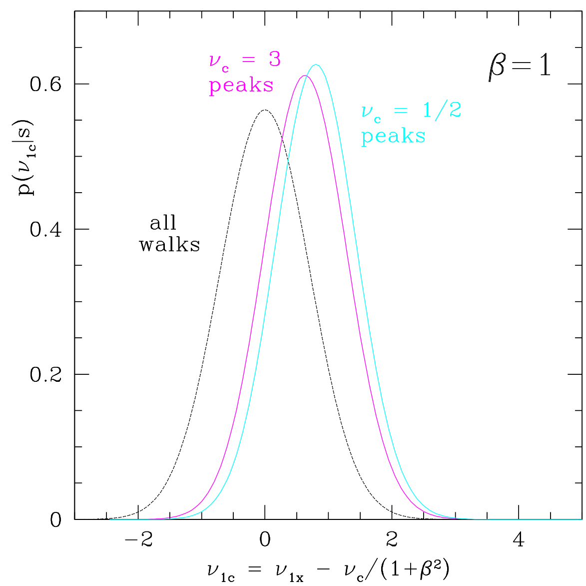

Figure 4. Distribution of walk height at first crossing, $\delta_{1\times}$, for all walks (dotted) and for walks which are also peaks in δ on scale s (solid); i.e., equations (9) and (36) respectively. The distribution for peaks in $\delta - g$ is also given by the dotted curve.

where $\nu_\beta \equiv (\delta_c/\sigma)/\sqrt{1+\beta^2}$ as before (c.f. equation 7), the parameters γ and R_* are defined by equation (4.6a) in Bardeen et al. (1986),

$$p(x|\gamma,\nu_\beta) = \frac{\mathrm{e}^{-(x-\gamma\nu_\beta)^2/2(1-\gamma^2)}}{\sqrt{2\pi(1-\gamma^2)}} \qquad (33)$$

is the usual conditional Gaussian (i.e. γ is the correlation coefficent between x and ν_β), and $\mathcal{F}(x)$ is given by equation (A15) of Bardeen et al. (1986). (The Musso-Sheth approximation for the first crossing distribution for all walks with correlated steps has $\mathcal{F}(x) = 1$ and $R = R_*$.)

The distribution of $\delta_{1\times}$ is then unchanged from that for all walks (equation 9), because a constraint on the 'velocity' of g_-, which is what the peaks constraint boils down to (Musso & Sheth, 2012), means nothing for g_+, which is what determines $\delta_{1\times}$. The statistics of walks centered on a randomly chosen particle within a protohalo are known to be different from those centered on the protohalo center of mass; the latter yield larger values of $\delta_{1\times}$ (Sheth et al., 2001; Achitouv et al., 2013; Despali et al., 2013). Therefore, the analysis above indicates that a model which identifies protohalo centers of mass with peaks in $\delta - g$ cannot explain this difference.

If we identify protohalo centers of mass on scale s with positions where δ first exceeds $\delta_c + g$ and are peaks in δ (rather than in $\delta - g$) on that scale, then the first crossing distribution becomes

$$sf(s) = \frac{\exp(-\nu_\beta^2/2)}{2\gamma\,(R_*/R)^3\sqrt{2\pi}} \int \mathrm{d}x\, x\, p(x|\gamma,\nu_\beta)\, G_0(x,\gamma_\beta x) \qquad (34)$$

where we have defined $\gamma_\beta^2 \equiv (1+\beta^2)^{-1}$, ν_β, γ, R_* and $p(x|\gamma,\nu_\beta)$ were defined above, and

$$G_n(x,\gamma_\beta x) \equiv \int dy\, p(y|\gamma_\beta, x)\,\mathcal{F}(y)\, y^n \qquad (35)$$

with $\mathcal{F}(y)$ the same quantity that appears in equation (32), i.e., given by equation (A15) of Bardeen et al. (1986). Similarly, a little algebra shows that in this case the distribution of $\delta_{1\times}$ is given by

$$p_{\rm pk}(\delta_{1\times}|s) = A_{1\times}\, p(\delta_{1\times}|s)\, \left[G_1 - \gamma(\nu_{1\times} - \nu_c)\, G_0\right], \quad (36)$$

where $p(\delta_{1\times}|s)$ is the distribution for all walks (equation 9), $G_n(\nu_{1\times}, \gamma\nu_{1\times})$ is given by equation (35), and $A_{1\times} = [\sqrt{1+\beta^2}\int \mathrm{d}x\, x\, p(x|\gamma,\nu_\beta)\, G_0(x,\gamma_\beta x)]^{-1}$ is a normalization factor which ensures that the integral over all $\delta_{1\times}$ yields unity.

In the limit $\beta \to 0$, the distribution $p(y|\gamma_\beta, x)$ becomes sharply peaked around its mean value $\gamma_\beta x \to x$, so that $G_0(x,\gamma_\beta x) \to \mathcal{F}(x)$. Thus, in this limit, equation (34) reduces to equation (32). Similarly, $p(\delta_{1\times}|s)$ becomes a delta function centered on $\delta_c/(1+\beta^2)$, making $p_{\rm pk}(\delta_{1\times}|s) \to A_{1\times}\, p(\delta_{1\times}|s)\, G_1$. Since $A_{1\times} \to G_1^{-1}$ in this limit, $p_{\rm pk}(\delta_{1\times}|s) \to p(\delta_{1\times}|s)$ as it should.

In general, at large $\nu_{1\times}$, $G_1/G_0 \to \gamma\nu_{1\times}$ making $p_{\rm pk}(\delta_{1\times}|s) \propto p(\delta_{1\times}|s)\, G_0(\nu_{1\times}, \gamma\nu_{1\times})$; this illustrates that the term in square brackets acts to skew the distribution towards larger $\delta_{1\times}$. Figures 3 and 4 show this explicitly: they compare $sf(s)$ and $p(\delta_{1\times}|s)$ for these two peak models with that for all walks. In practice, we use equations (4.4) and (6.13) of Bardeen et al. (1986) to approximate G_0 and G_1/G_0, and we assumed Gaussian smoothing of a scale-free power spectrum, i.e. $P(k) \propto k^n$, for which $\gamma^2 = (n+3)/(n+5)$ and $(R_*/R)^2 = 6/(n+5)$. To

make the Figures, we set $n = -1.2$ and $\beta = 1$ to highlight the effects of β.

Figure 3 shows that peaks in δ and $\delta - g$ do indeed produce different counts (short and long dashed curves, respectively); both are different from the result for all walks (dotted). And Figure 4 shows that the distribution given in equation (36) is indeed shifted to larger values of $\delta_{1\times}$, with the shift depending weakly on the mass scale ν_c. This increase in $\delta_{1\times}$ is qualitatively in the right direction, suggesting that identifying protohalos with peaks in δ is a better model than one where protohalos are identified with peaks in $\delta - g$. However, the predicted distribution for peaks is not as different from that for all walks as is the difference seen in simulations between centre-of-mass walks and randomly chosen ones (the shift in the mean is not large enough, the width is not narrow enough, and the shape is not skewed enough).

Before moving on, we note that, in the one-dimensional problem, the peaks motivated approach is attractive because it provides a natural reason why halo counts in simulations do not fall as steeply as $\exp(-\delta_c^2/2s)$ at small s. The two-Gaussian model here achieves this by setting $\beta \approx 0.6$ (see discussion at end of Section 2.2). The analysis above indicates that peaks in this two-Gaussian model will require a smaller value of β to reproduce the halo counts. Then reproducing the distributions of $p(\delta_{1\times}|s)$ *and* $p_{\rm pk}(\delta_{1\times}|s)$ provide important self-consistency tests. Since reducing β from the value used to make Figure 4 will only make all the curves there more similar to one another, this will exacerbate the discrepancies between model and simulations. Thus, our analysis suggests that neither of the peaks models we have considered here are consistent with measurements.

4 DISCUSSION

We described a two-dimensional excursion set model, for Gaussian walks in δ and g, for which almost all quantities associated with first crossing distributions can be computed analytically. We have tested all the analytic expressions we provide in this paper using Monte-Carlo realizations of the two-dimensional stochastic process, finding excellent agreement. Since the analytic arguments are sufficiently simple, we have only included a few plots showing this agreement.

Our predictions include the unconditional first crossing distribution $f(s|z)$ (Section 2.2); the conditional first crossing distribution for redshift z, $f(s, z|S, Z)$, by walks which are known to have first crossed one another on scale $S < s$ at redshift $Z < z$ (Section 2.5); and the conditional distribution $f(s, z|S, \Delta)$ for walks which are constrained to have height Δ on scale $S < s$ (Section 2.6). These are usually used to model halo abundances, progenitor distributions, and the environmental dependence of clustering. In the one-dimensional case, for appropriately chosen pairs of redshift and environment, the progenitor and conditional distributions are the same. For higher-dimensional walks, this is no longer the case: the conditional distributions generically predict more massive objects (Figure 1 and related discussion).

Another new feature of such higher-dimensional models is the fact that there is, generically, a distribution of walk heights at first crossing $p(\delta_{1\times}|s)$ (Section 2.3), and an associated distribution of the other variable $p(g_{1\times}|s)$ (Section 2.4). For the Gaussian walks considered here, these distributions are Gaussian, even when the barrier height depends on the first crossing scale s. We argued that s-dependence of the mean barrier height, with a Gaussian scatter around the mean, should provide a good approximation even when the walks are not Gaussian (Section 3.3).

We also argued that, because of the variable(s) which are not δ, halo bias in these models will generally be stochastic (sometimes refered to as nonlocal), and the conditional distributions will generically exhibit assembly bias, even when the steps in the walks are uncorrelated. We provided explicit expressions for both the stochastic (Section 2.7) and the assembly bias (Section 2.8 and Figure 2). Although our model predicts no correlation between halo formation times and environment (at fixed halo mass), in agreement with the one-dimensional case, it nevertheless predicts that halos surrounded by overdensities should be denser and more concentrated than halos of the same mass in underdensities.

The lack of correlation between time and environment is a consequence of studying walks with uncorrelated steps. We sketched how to generalize our results to include correlations between the steps in each walk (necessary for quantitative comparison with simulations; Section 3.1), and between the walks themselves (as might arise in models where the critical density required for collapse is determined by the overdensity on large scales; Section 3.2). These will introduce additional assembly bias effects, for the same reasons they do so for one-dimensional walks. Although we sketched how to quantify these here, we did not show plots or otherwise quantify these effects for the following reason.

One of the drawbacks of this model – that is in common with the usual one-dimensional walk approach – is that it is explicitly about the statistics of all points in space. However, halos form around special positions in space, and the statistics of this point process – arguably the point process for which the description of the physics is simplest – is very different from that around randomly chosen positions (Sheth et al., 2001; Paranjape & Sheth, 2012; Achitouv et al., 2013). We argued that that the simplest case, in which halos form around positions which are peaks in $\delta - g$, cannot explain this difference (Section 3.4). Although a model in which halos form around peaks in δ fares better (Figures 3 and 4), it fails to adequately model the differences between walks centred on all particles, and those centred on the special subset which are protohalo centers of mass. Work in progress shows how to extend this approach to include a more elaborate model for protohalo centers-of-mass, but we believe our results demonstrate the power of requiring

models to reproduce not just halo counts but the distribution of δ at fixed halo mass as well.

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APPENDIX A: PROOF OF EQUATION (15)

The main complication with respect to the one dimensional case is that the constraint that the walk passed through Δ on scale S still allows walks with a range of values of G. This range is constrained by the requirement that Δ and G had not crossed on scales smaller than S. At fixed Δ *and* G, the solution is straightforward, as we show shortly, so the main work is to integrate this solution over the allowed range of G.

As before, it is best to work in the (g_+, g_-) plane, in which case the requirement that the walk has height Δ on scale S means that

$$G_- < \frac{\delta_c(S)}{\sqrt{1+\beta^2}} \quad \text{and} \quad G_+ = \Delta\sqrt{1+\beta^2} - G_- \tag{A1}$$

(again capital letters indicate values at S). The distribution of s at which $\delta_c(s)/\sqrt{1+\beta^2}$ is first crossed, given that the walk started from (G_+, G_-) on scale S, is given by equation (6) with

$$\nu^2 = \frac{(\delta_c/\sqrt{1+\beta^2} - G_-)^2}{s-S}. \tag{A2}$$

Notice that this expression depends only on G_-, so we will denote the associated first crossing distribution as $f(s|G_-, S)$.

To get the quantity we are after, $f(s|\Delta, S)$, we must now integrate $f(s|G_-, S)$ over all allowed starting values (G_+, G_-), weighting by the probability of starting at each. I.e.,

$$f(s|\Delta,S) = A\int_{-\infty}^{\infty} \mathrm{d}G_+ \int_{-\infty}^{\delta_c/\sqrt{1+\beta^2}} \mathrm{d}G_-\, f(s|G_-,S) \times \; p(G_+|S)\, q(G_-|S)\, \delta_{\rm D}(\delta-\Delta) \tag{A3}$$

where

$$q(G_-|S) = \frac{\mathrm{e}^{-G_-^2/2S}}{\sqrt{2\pi S}} - \frac{\mathrm{e}^{-(2\delta_c/\sqrt{1+\beta^2}-G_-)^2/2S}}{\sqrt{2\pi S}} \tag{A4}$$

is the probability that (the one-dimensional) walk g_- has height G_- at S and never crossed $\delta_c/\sqrt{1+\beta^2}$ on some smaller $s < S$ (Bond et al., 1991), $p(G_+|S)$ is a Gaussian with zero mean and variance S, and

$$A \equiv \int_{-\infty}^{\infty} \mathrm{d}G_+ \int_{-\infty}^{\delta_c/\sqrt{1+\beta^2}} \mathrm{d}G_-\, p(G_+|S)\, q(G_-|S)\, \delta_{\rm D}(\delta-\Delta) \tag{A5}$$

is a normalization constant which ensures that the probabilities integrate to unity. This, and the integral in eq.(A3) can be performed analytically, yielding

$$f(s|\Delta,S) = \frac{(\delta_c-\Delta)(1+\beta^2)}{s-S+\beta^2 s}\,\frac{\exp^{-(\delta_c-\Delta)^2/2(s-S+\beta^2 s)}}{\sqrt{2\pi(s-S+\beta^2 s)}}. \tag{A6}$$

This is equivalent to the change of variables given by equation (15) of the main text.